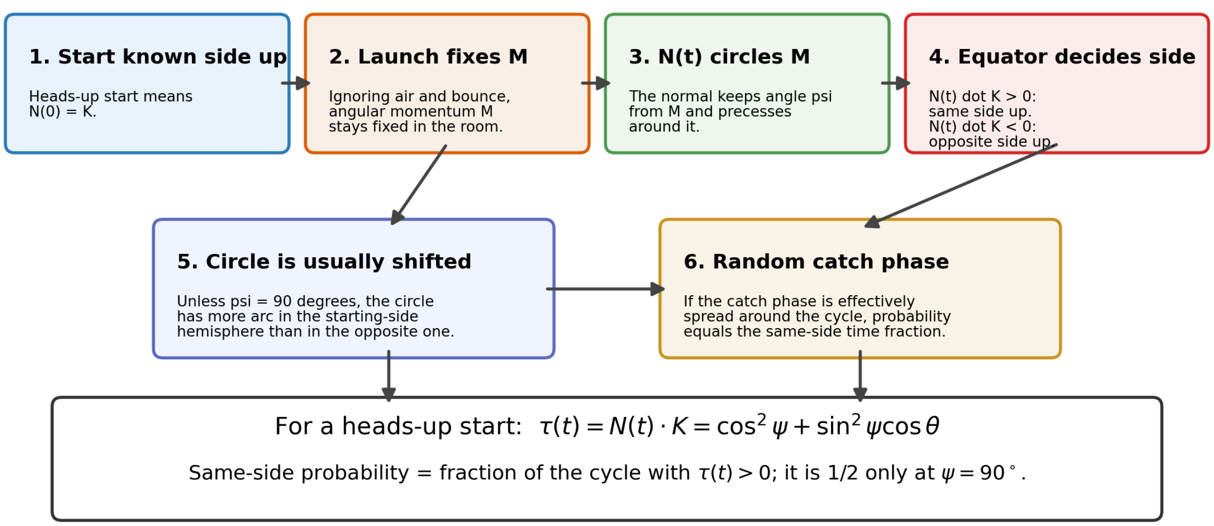
Why a Flipped Coin Favors the Side It Started On

A visual, plain-language tutorial on precession in the coin-toss model

Based on the model in *Dynamical Bias in the Coin Toss* by Persi Diaconis, Susan Holmes, and Richard Montgomery

Putting the argument together



What this document explains. A coin does not “know” whether it began heads-up or tails-up. The same-side bias comes from geometry: the coin’s normal vector moves on a precession circle that is usually shifted toward the starting-side hemisphere.

Companion simulation: `coin_precession_sim_v2.html`. It is self-contained and can be opened locally in a browser.

Contents

The whole story in one page

The model is easiest to understand if we stop looking at the whole coin and instead track one arrow: the arrow sticking out of the starting face. If the coin starts heads-up, this arrow points straight up at the start.

Diaconis, Holmes, and Montgomery show that, while the coin is in flight and idealized effects such as air resistance and bouncing are ignored, the angular momentum vector M stays fixed. The coin's normal vector $N(t)$ keeps a constant angle ψ from M and precesses around it. Heads/tails is determined by the sign of $N(t) \cdot K$, where K is straight up.

For a coin started exactly heads-up, the model gives

$$\tau(t) = N(t) \cdot K = \cos^2 \psi + \sin^2 \psi \cos(\omega_N t).$$

If $\tau(t) > 0$, the starting side is up. If $\tau(t) < 0$, the opposite side is up.

The crucial term is the positive offset $\cos^2 \psi$. Unless $\psi = 90°$, the curve is shifted upward. That means the coin spends more of its precession cycle with the starting side up than with the opposite side up.

The shortest possible explanation. The coin does not remember a label such as "heads." What persists is a component of the starting normal vector along the fixed angular momentum vector M. This preserved component shifts the normal's path toward the side that was initially up.

For vigorous flips, the catch phase is treated as effectively spread around the precession cycle. Then the probability of landing the same way it started is the fraction of the cycle for which $\tau(t) > 0$. In the paper's idealized formula,

$$p(\psi) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}\left(\cot^2 \psi\right)$$

when $45° < \psi < 135°$, with $p(\psi) = 1$ outside that middle range. This equals $1/2$ only at $\psi = 90°$.

Symbols and vocabulary

Symbol	Meaning
K	The fixed upward direction in the room. It is not attached to the coin.
$N(t)$	The normal vector: an imaginary arrow sticking straight out of the heads face at time t. If the coin starts heads-up, $N(0) = K$.
M	The angular momentum vector: the fixed "rotation-summary" arrow during ideal flight. For a simple wheel, it points along the axle.
ψ	The starting angle between $N(0)$ and M. The ideal end-over-end fair case has $\psi = 90°$.
θ	The phase around the precession circle. You can think of it as "where the normal is in the cycle."
$\tau(t)$	The upwardness score $N(t) \cdot K$. Positive means starting side up; negative means opposite side up.

Two warnings about language

First, "precession" here means that the normal vector $N(t)$ moves around the angular momentum vector M. It is not a mystical tendency toward heads. It is ordinary rigid-body geometry.

Second, the bias is a *same-side* bias, not a heads bias. If the coin starts heads-up, the bias favors heads. If it starts tails-up, the same mechanism favors tails.

Step 1: three arrows, not a whole coin

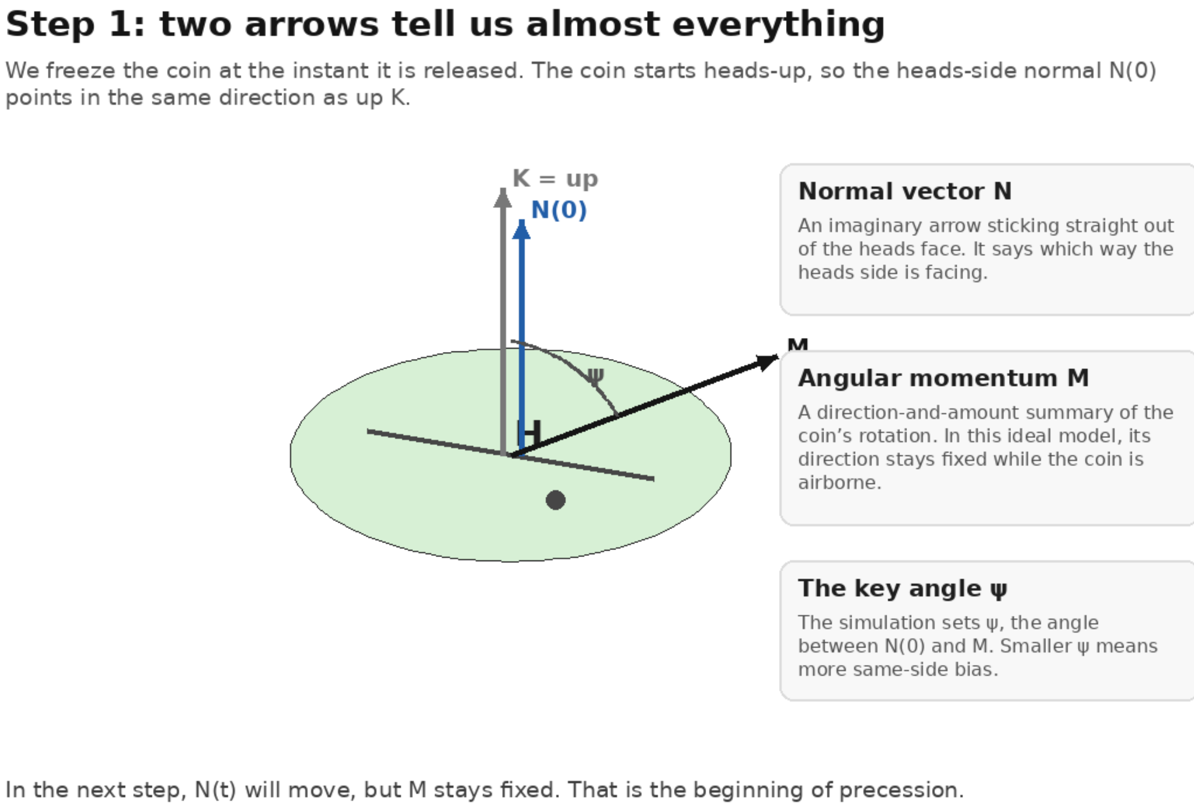


Figure 1: The basic ingredients: room-up K, the heads-normal $N(0)$, and angular momentum M.

Start with three arrows:

- K means straight up in the room.
- $N(t)$ is the normal vector: the arrow sticking out of the heads face of the coin.
- M is the angular momentum vector, the rotation-summary arrow fixed by the launch.

If the coin starts heads-up, then the heads-normal starts straight up:

$$N(0) = K.$$

The coin's future orientation is described by how $N(t)$ moves relative to K.

Takeaway. K is room-up. $N(t)$ is the arrow out of the heads face. M is the fixed rotation-summary arrow.

Step 2: angular momentum is the fixed rotation-summary arrow

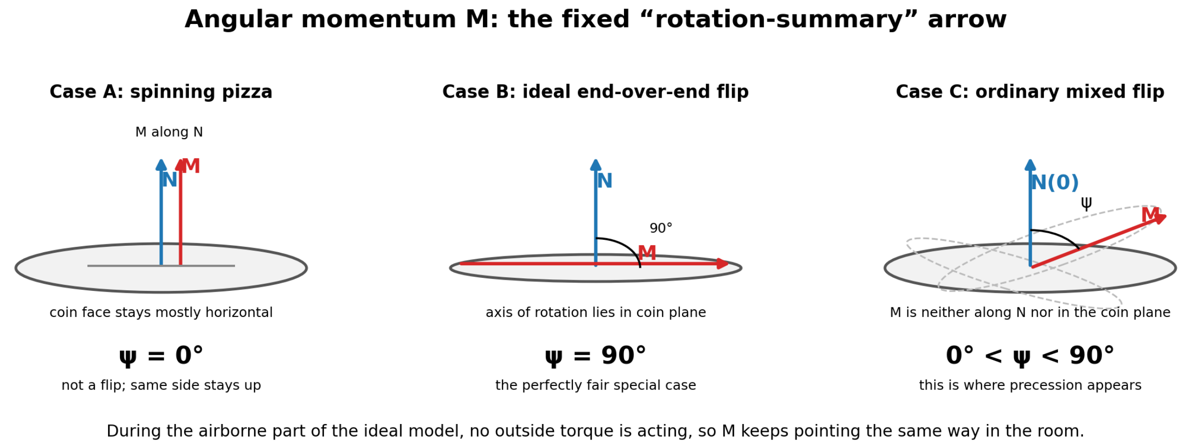


Figure 2: Three clean cases: pizza spin, ideal end-over-end flip, and a mixed real flip.

A gentle way to think of angular momentum is:

$$\text{angular momentum} \approx \text{the arrow that summarizes spin.}$$

For a simple spinning wheel, it points along the axle. For a coin in ideal free flight, the important fact is that M stays fixed in the room.

Three cases matter:

- **Pizza spin:** M points along N. Then $\psi = 0°$ and the coin stays the same side up.
- **Ideal end-over-end flip:** M lies in the plane of the coin. Since N is perpendicular to the face, $\psi = 90°$.
- **Ordinary mixed flip:** M is not exactly along N and not exactly in the coin's plane. Then $0° < \psi < 90°$.

Takeaway. The same-side effect lives in the ordinary mixed case: the coin is partly flipping end-over-end and partly spinning about its own face.

Step 3: why the normal traces a circle

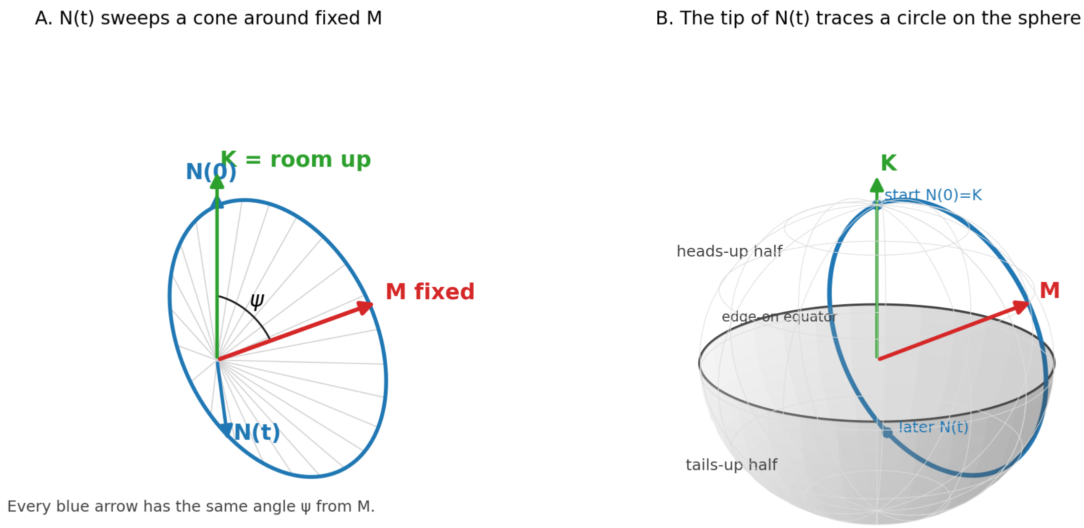


Figure 3: $N(t)$ keeps the same angle from M, so it sweeps a cone; the tip of the arrow traces a circle on the sphere.

During ideal flight, two things happen at once:

$$M \text{ stays fixed,} \qquad \text{and} \qquad \angle(N(t), M) = \psi \text{ stays fixed.}$$

If an arrow is forced to keep the same angle from a fixed pole, its tip moves around a circle. That is the precession circle.

This is the key geometry. The coin is not deciding anything. The normal vector is constrained to move along a circle centered around M.

Takeaway. Precession means: $N(t)$ goes around the fixed angular momentum vector M.

Step 4: heads/tails is an equator test

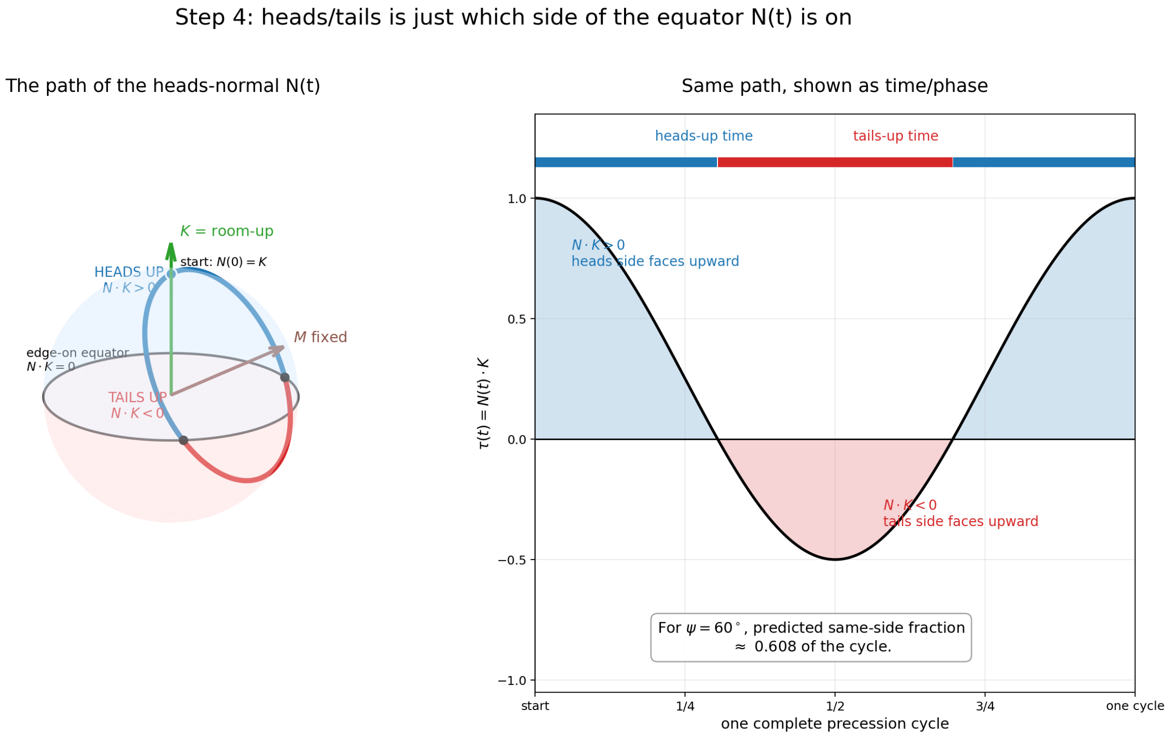


Figure 4: The upper hemisphere is starting-side-up; the lower hemisphere is opposite-side-up.

Heads-up or tails-up is decided by the sign of one dot product:

$$\tau(t) = N(t) \cdot K.$$

This is the upwardness score of the heads face.

- If $N(t) \cdot K > 0$, the heads-normal points partly upward, so the heads face is up.
- If $N(t) \cdot K = 0$, the coin is edge-on.
- If $N(t) \cdot K < 0$, the heads-normal points partly downward, so the opposite face is up.

If the coin starts heads-up, the blue parts of the path are starting-side-up and the red parts are opposite-side-up. In the example $\psi = 60°$, the starting-side arc is visibly longer.

Takeaway. The coin is starting-side-up exactly when $N(t)$ lies in the upper hemisphere.

Step 5: why the starting-side arc is longer

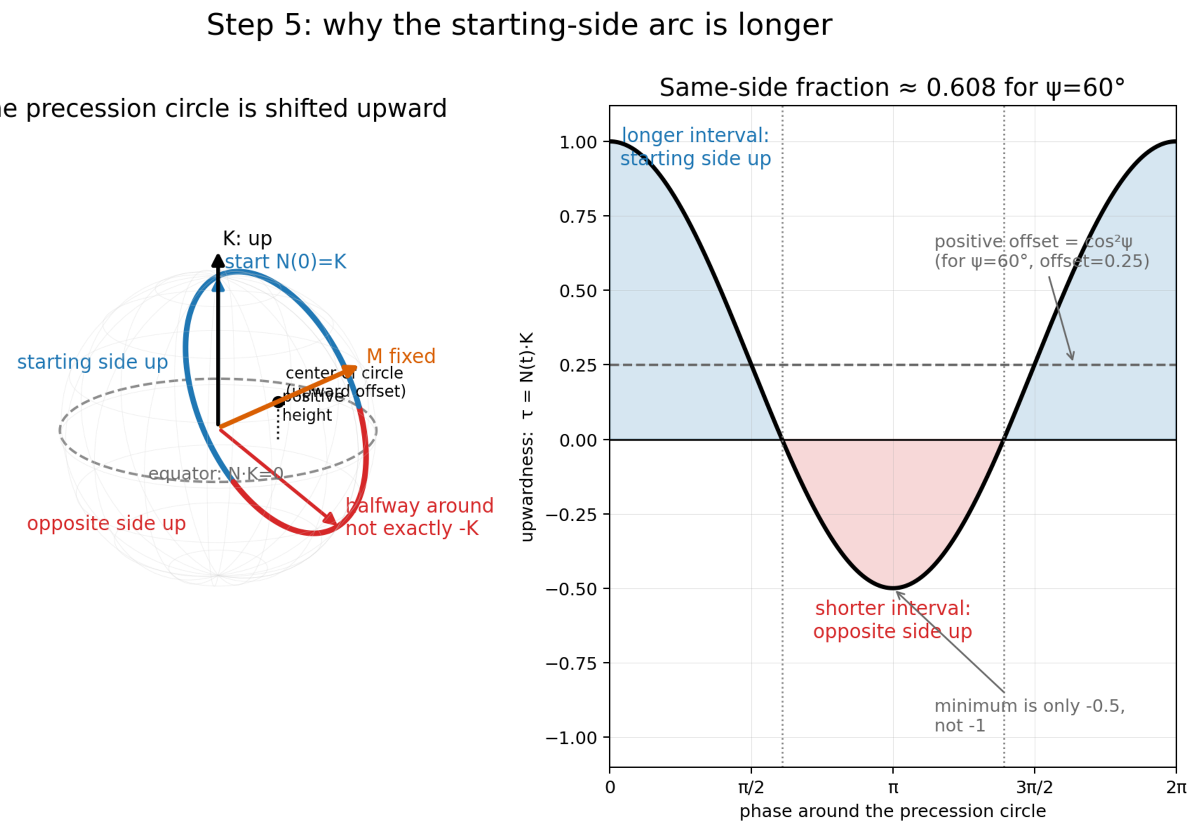


Figure 5: For $\psi = 60°$, the upwardness curve is shifted upward, so it is positive more often than negative.

Here is the heart of the matter: the precession circle is usually shifted toward the starting side.

For a heads-up start, the paper's formula becomes

$$\tau(t) = \cos^2\psi + \sin^2\psi\cos\theta,$$

where θ is the phase around the precession circle. The oscillating term $\sin^2\psi\cos\theta$ goes up and down. But the term $\cos^2\psi$ is a positive offset.

For $\psi = 60°$,

$$\tau(t) = 0.25 + 0.75\cos\theta.$$

The curve is negative only when $\cos\theta < -1/3$. That happens for about 39.2% of the cycle, so the starting side is up for about 60.8% of the cycle.

Takeaway. The coin does not remember a label. The normal's circular path is usually off-center, and the off-center circle spends more time in the starting-side hemisphere.

Step 6: the special fair case $\psi = 90°$

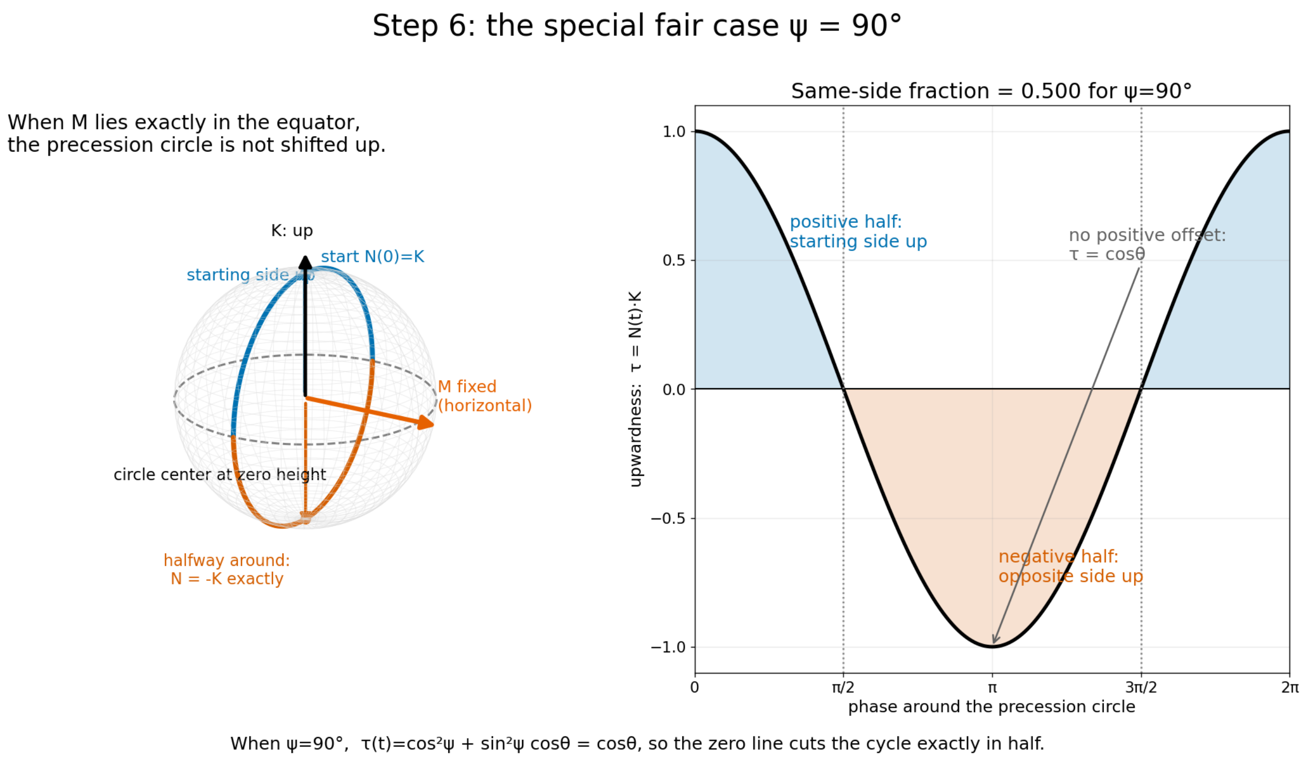


Figure 6: When $\psi = 90°$, the circle is centered symmetrically: half starting-side-up and half opposite-side-up.

The ideal end-over-end flip has

$$\psi = 90°.$$

In this case, M lies exactly in the plane of the coin. The precession circle is not shifted upward or downward; it passes symmetrically through the north and south poles.

Plugging $\psi = 90°$ into the formula gives

$$\tau(t) = \cos\theta.$$

That curve is positive half the time and negative half the time. So the same-side fraction is exactly

$$0.5.$$

Takeaway. $\psi = 90°$ is the special fair case. The positive offset disappears.

Step 7: real human flips are near fair, but not exactly fair

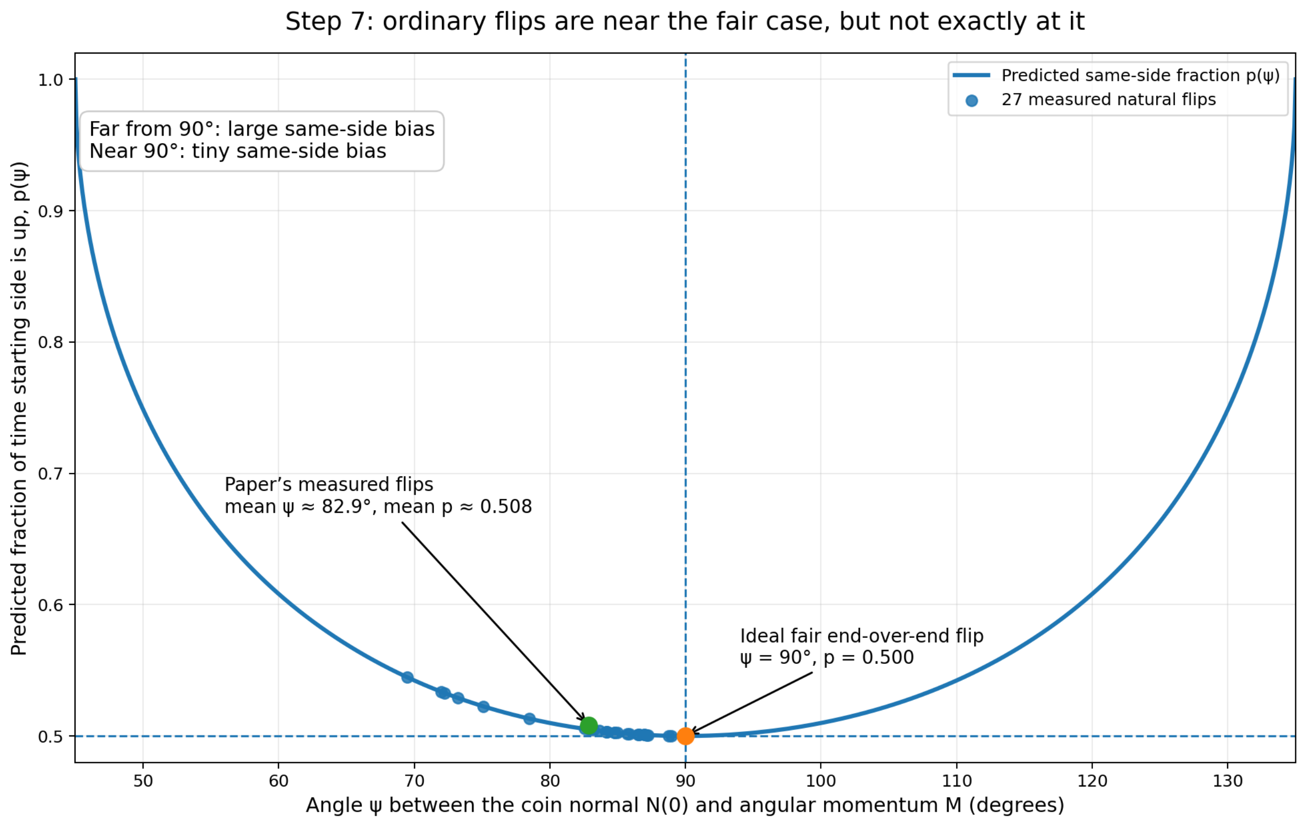


Figure 7: The predicted same-side fraction is close to 0.5 when ψ is close to 90°. The measured flips in the paper cluster near this fair point.

A real human flip is often mostly end-over-end, but not perfectly. The thumb usually gives the coin a little spin about its own face too. That means ψ is near, but not exactly, 90°.

The effect is small near 90°:

$$\psi = 90^\circ \Rightarrow p = 0.500,$$

$$\psi = 85^\circ \Rightarrow p \approx 0.502,$$

$$\psi = 80^\circ \Rightarrow p \approx 0.510.$$

This is why the effect is hard to notice casually. The paper's measured natural flips gave an average predicted same-side probability of about 0.51.

Takeaway. Ordinary flips are close to the fair case, so the same-side bias is usually tiny rather than dramatic.

Step 8: same-side bias is not heads bias

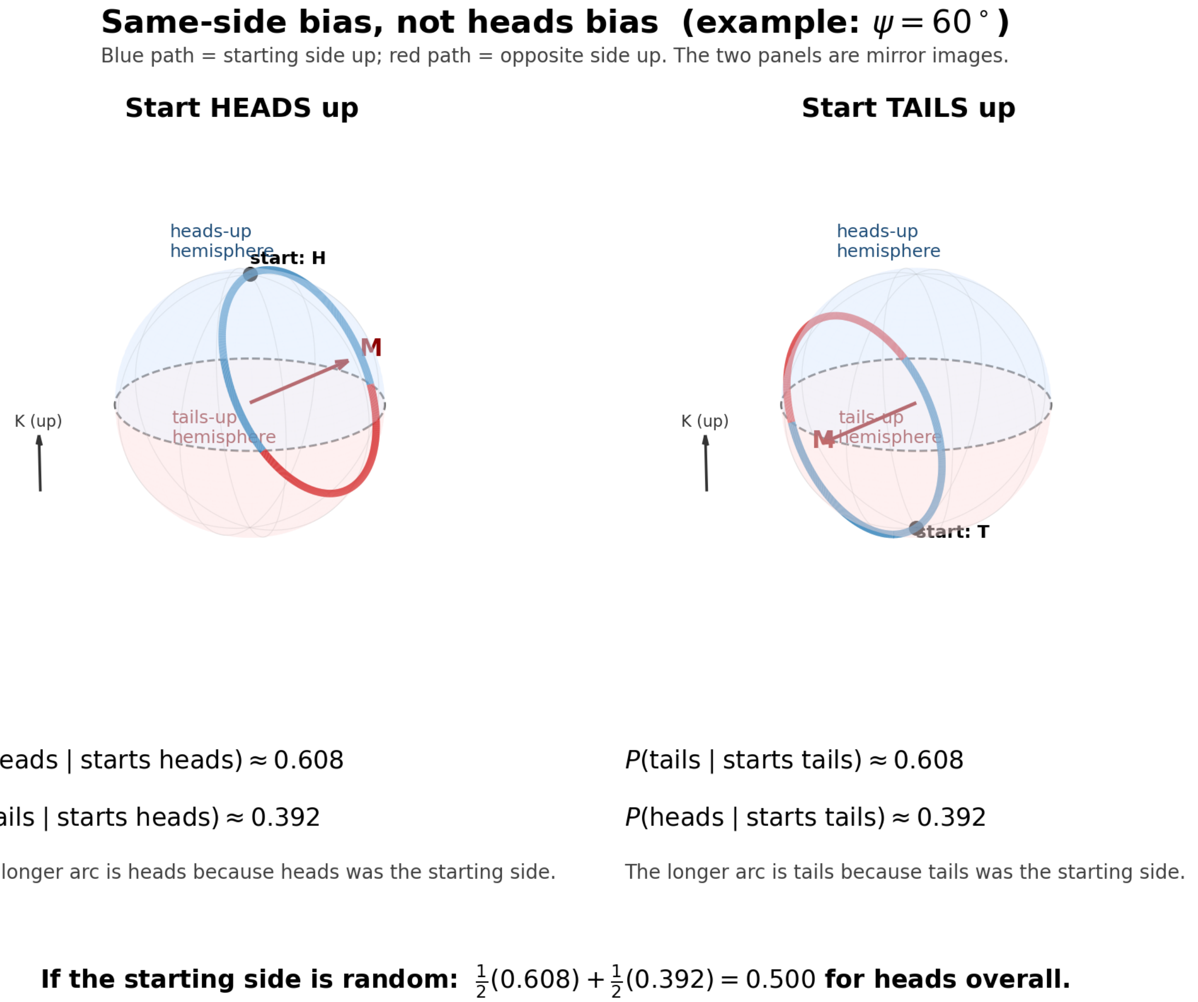


Figure 8: If the coin starts heads-up, same-side means heads. If it starts tails-up, same-side means tails.

Let p be the probability that the coin lands the same way it started.

If the coin starts heads-up, then

$$P(\text{heads at end} \mid \text{starts heads}) = p.$$

If the coin starts tails-up, then same-side means tails, so

$$P(\text{heads at end} \mid \text{starts tails}) = 1 - p.$$

If the starting side is random, half heads and half tails, then

$$P(\text{heads}) = \frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}.$$

Takeaway. The model favors continuity of orientation. It does not favor heads as such.

Step 9: why time fraction becomes probability

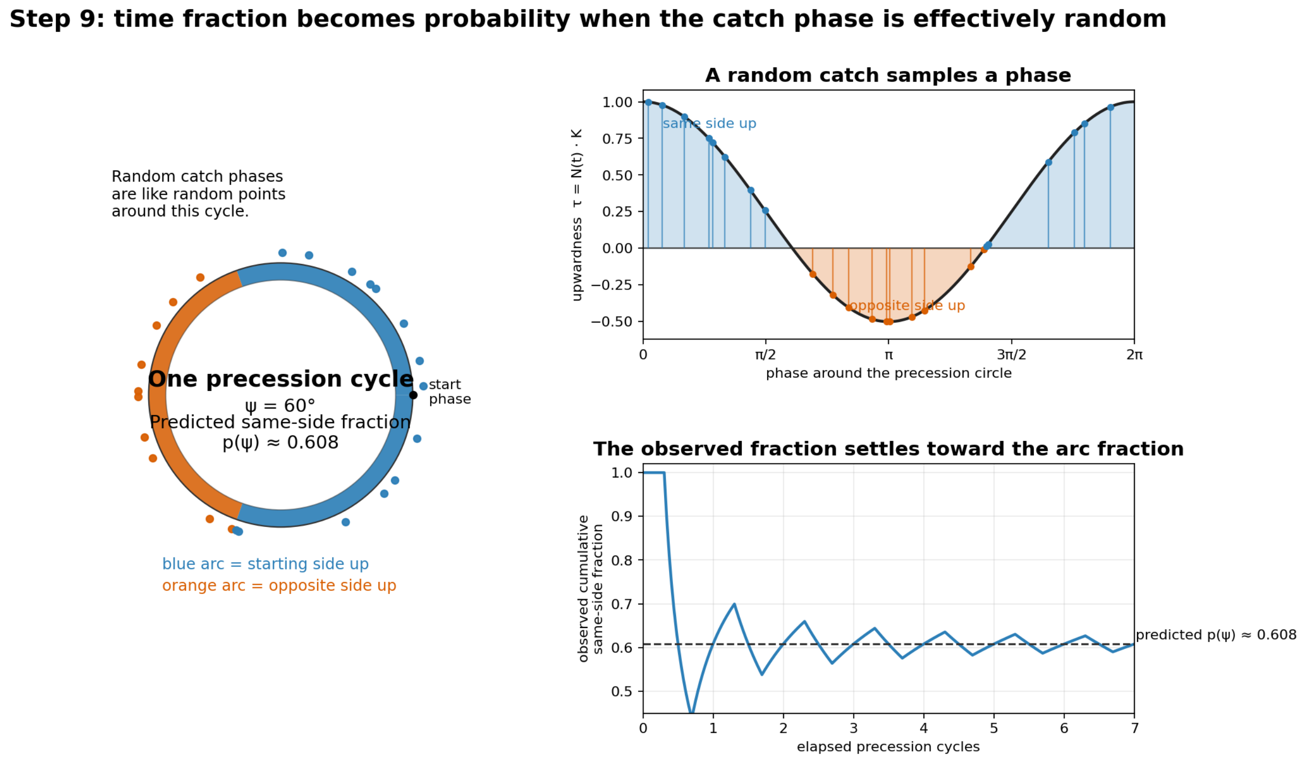


Figure 9: A random catch phase is like picking a random point around the precession cycle. The running observed fraction settles toward the predicted time fraction.

A single exact toss is deterministic: if you knew the exact launch and catch time, the result would be fixed.

Probability enters because the catch phase is uncertain. The phase

$$\theta = \omega_N t \pmod{2\pi}$$

means where $N(t)$ is around the precession circle when the coin is caught.

For a vigorous flip, small changes in launch and catch timing can move the coin to very different phases. The paper models this by treating the catch phase as effectively spread around the cycle. Then the probability of seeing the starting side equals the fraction of the precession cycle spent starting-side-up.

The lower-right panel in the figure shows exactly what the companion simulation displays: the cumulative observed same-side time fraction converges toward the predicted same-side fraction.

Takeaway. If the catch phase is effectively random, probability equals time fraction.

Step 10: the compact picture

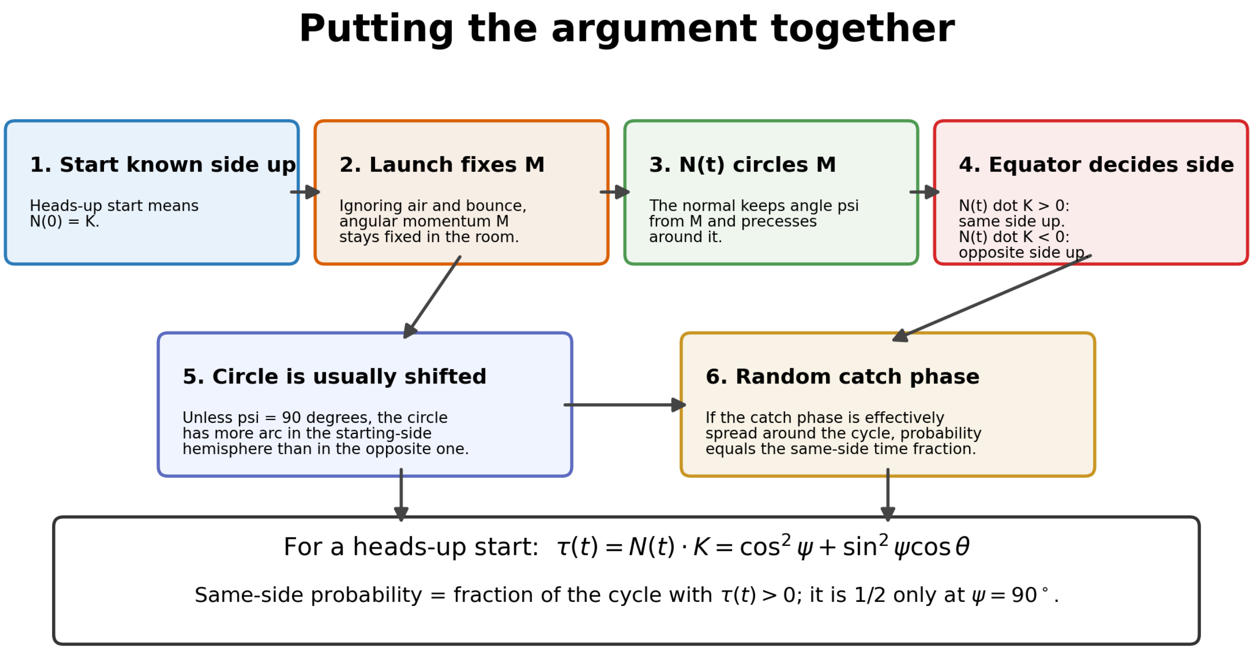


Figure 10: The complete chain of reasoning.

The intuition can now be compressed into one sequence:

1. The starting face determines the starting normal: for heads-up, $N(0) = K$.
2. The launch fixes an angular momentum vector M.
3. During ideal flight, M is fixed and $N(t)$ precesses around M at a fixed angle ψ.
4. The sign of $N(t) \cdot K$ decides which side is up.
5. Unless $\psi = 90°$, the circle traced by $N(t)$ is shifted toward the starting side.
6. If the catch phase is effectively random, the probability of same-side landing is the fraction of the circle in the starting-side hemisphere.

The answer to "how does the coin know?" It does not know. The geometry preserves a component of the starting normal along the fixed angular momentum vector. That preserved component shifts the whole precession circle toward the starting side.

What the companion simulation is doing

The simulation is intentionally simpler than a full coin toss. It does not simulate the thumb, air resistance, the parabolic center-of-mass path, the catch, or bouncing. Instead, it isolates the orientation geometry:

$$K = (0, 0, 1),$$

$$M = (\sin\psi, 0, \cos\psi),$$

$$N(t) = \text{rotate } K \text{ around axis } M \text{ by phase } \theta.$$

Then it checks the sign of

$$N(t) \cdot K.$$

The simulation's running curve is the cumulative same-side time fraction:

$$\frac{\text{time so far with } N(t) \cdot K > 0}{\text{time elapsed so far}}.$$

The horizontal predicted line is the theoretical same-side fraction from the formula. When $\psi = 60^\circ$, that line is about 0.608; when $\psi = 90^\circ$, it is 0.5.

How to run it. Save `coin_precession_sim_v2.html` next to this PDF or anywhere on your computer. Open it by double-clicking. It uses no internet connection and no external libraries.

Mini-FAQ

If the coin is flipped without precession, is $\psi = 90°$?

For the ordinary ideal end-over-end flip, yes. The angular momentum vector lies in the coin's plane, so it is perpendicular to the normal. That gives $\psi = 90°$.

But "without precession" can also describe a pizza-spin case where M points along N, giving $\psi = 0°$. That case is not really a flip; the same side stays up.

Why does "halfway around" not mean exactly upside down?

Because the coin is halfway around the precession circle, not necessarily halfway around a horizontal flip axis. Unless M is exactly horizontal, the part of $N(t)$ along M remains. So the opposite point of the precession circle is usually not exactly $-K$.

Why does the effect disappear at $\psi = 90°$?

At $\psi = 90°$, the positive offset $\cos^2 \psi$ is zero. The upwardness curve is simply $\cos \theta$, which is positive half the time and negative half the time.

Does this mean coin tossing is unfair?

It means that, in the ideal caught-coin model, a known starting side is slightly favored. It does not mean heads is intrinsically favored. If the starting side is random, the heads probability remains 1/2.

What assumptions matter?

The main tutorial model ignores air resistance and bouncing, assumes the coin is caught, and assumes we know the starting orientation. The paper discusses these caveats explicitly. For ordinary caught flips, the predicted same-side bias is small.

Reference and source note

The explanation in this tutorial is based on:

Persi Diaconis, Susan Holmes, and Richard Montgomery, “Dynamical Bias in the Coin Toss,” *SIAM Review*, 49(2), 211–235, 2007.

Key elements used here:

- The formula $\tau(t) = \cos^2\psi + \sin^2\psi\cos(\omega_N t)$ for a heads-up start.
- The sphere picture: $N(t)$ traces a circle; the equator separates starting-side-up from opposite-side-up.
- The same-side probability formula $p(\psi)$ for vigorous flips.
- The empirical estimate that ordinary natural flips give a same-side probability around 0.51.

All tutorial figures are newly generated illustrations of the model, not reproductions from the paper.